Perspectivity, special cleanness and chains of idempotents

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The case of unit-regular rings

Theorem (Fuchs, Kaplansky, Handelman, Camillo, Khurana)

Let R be a ***regular*** *ring. Then the following statements are equivalent:*

1. *R is unit-regular;*
2. *isomorphic idempotents have isomorphic complements;*
3. *elements of R are special clean;*
4. *R has stable range 1;*
5. *R is perspective;*
6. *$\mathcal{M}_2(R)$ has perspectivity transitive.*

General case?

Objective:

We consider the interplay between the following notions in the general setting:

1. **Perspectivity** of direct summands;
2. element-wise properties of **regular elements** (or endomorphisms);
3. Relations between **Idempotents**.

(Some) Questions:

1. Is there an element-wise characterization of perspectivity?
2. Is there a characterization based on idempotents?
3. Can we characterize rings where all regular elements are special clean?
4. If R is IC and perspectivity is transitive, is R perspective?

Perspectivity in modules

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- The module M is *perspective* if for any two $A, B \subseteq^{\oplus} M$, $A \simeq B \Rightarrow A \sim_{\oplus} B$;
- The module M is *3/2-perspective* if for any two $A, B \subseteq^{\oplus} M$ and any complementary summand $\bar{A}$ of A, $A \simeq B \Rightarrow \bar{A} \sim_{\oplus} \bar{B}$ for some complementary summand of B;

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- The module M has *perspectivity transitive* if $A \sim_{\oplus} B \sim_{\oplus} C \Rightarrow A \sim_{\oplus} C$.

Perspectivity and IC (internal cancellation)

M is IC if for any two $A, B \subseteq^{\oplus} M$ and $\bar{A}, \bar{B}$, $A \simeq B \Rightarrow \bar{A} \simeq \bar{B}$, that is

$$A \oplus \bar{A} = M = B \oplus \bar{B} \text{ et } A \simeq B \Rightarrow \bar{A} \simeq \bar{B}.$$

Any perspective module is $3/2$-perspective, and any $3/2$-perspective module is IC.

From modules to rings

Definition

A ring R is perspective (resp. $3/2$, IC) if the right module R_R is (iff ${}_RR$ is).

Theorem ("ER"-property)

M is perspective (resp. $3/2$, IC) iff the endomorphism ring $R = End(M)$ is.

Therefore, we can study R instead of M.

Regular and (special) clean elements, idempotents

- $a \in R$ is *regular* (resp. *unit-regular*) if $aua = a$ for some $u \in R$ (resp. u invertible). We note $reg(R)$ (resp. $ureg(R)$) the set of regular (resp. unit-regular) elements, and $V(a) = \{b \in R | aba = a, bab = b\}$ for the set of reflexive inverses of a;

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- a is *special clean* if $a = \bar{e} + u$ for some $u \in U(R)$ such that $e \in E(R)$ $aR \cap \bar{e}R = 0$, iff $a = \bar{e} + u = au^{-1}a$ for some $u \in U(R)$ and $e \in E(R)$;

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- $e, f \in E(R)$ are *isomorphic idempotents* if $eR \simeq fR$ iff $e = ab, f = ba$ for some $a, b \in R$ (and we can choose $aba = a, bab = b$);

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- $e, f \in E(R)$ are *isomorphic idempotents* if $eR \simeq fR$ iff $e = ab, f = ba$ for some $a, b \in R$ (and we can choose $aba = a, bab = b$);
- if $e \in E(R)$ we note $\bar{e} = 1 - e$ its complementary idempotent.

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Theorem

Let $a \in reg(R)$. Then the following statements are equivalent:

1. *(Def.) a is right perspective;*
2. *(aR, bR) For any $b \in V(a)$, aR and bR are perspective;*
3. *((Special) Clean) For any $f \in E(R)$ such that $Ra = Rf$, then $a = \bar{e} + u$ for some $u \in U(R), e \in E(R)$ such that $eR = fR$ (and the decomposition is actually special clean);*
4. *(Idempotent) for any $b \in V(a)$, there exists $e, g \in E(R)$ such that*

$$(ab)R = gR, Rg = Re \text{ and } eR = (ba)R.$$

Left-right symmetry of perspective elements

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Theorem (Left-right symmetry)

Right perspective elements and left perspective elements coincide.

(The proof actually relies on a result due to D. Khurana, P.P. Nielsen and X. Mary on chains of idempotents that will be defined shortly)

Example: Group invertible elements (in particular units or idempotents) are perspective.

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- $e \sim_r f$ if $ef = f, fe = f$ (e, f *right associates*);
- $e \sim_\ell f$ if $ef = e, fe = f$;
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Definition (Right n-chains)

Let $e, f \in E(R)$. e, f are *right n-chained* if
$e = g_0 \sim_r g_1 \sim_\ell \cdots g_n = f$ for some $g_1, \cdots, g_n \in R$.
For instance, e, f are right 3-chained if $e \sim_{r\ell r} f$.

We say that R satisfies $\mathcal{P}(n)$ if any two isomorphic idempotents are right n-chained.

We recall:

Theorem (Khurana, Lam)

The following statements are equivalent:

1. R *is IC;*
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It holds that:

Theorem

The following statements are equivalent:

1. R *is perspective;*
2. *Regular elements are perspective;*
3. *For any* $e, f \in E(R)$, $e \simeq f \Rightarrow e \sim_{r\ell r} f$ *(*R *satisfies* $\mathcal{P}(3)$*);*
4. *For any* $e, f \in E(R)$, $e \simeq f \Rightarrow \{e \sim_{r\ell r} f$ *or* $e \sim_{\ell r\ell} f\}$.

Theorem

The following statements are equivalent:

1. R *is* $3/2$*-perspective;*
2. *Regular elements are special clean;*
3. *For any* $e, f \in E(R)$, $e \simeq f \Rightarrow e \sim_{r\ell r\ell} f$ *(*R *satisfies* $\mathcal{P}(4)$*).*

We do not know if this is equivalent to the *a priori* weaker version: For any $e, f \in E(R)$, $e \simeq f \Rightarrow \{e \sim_{r\ell r\ell} f$ or $e \sim_{\ell r\ell r} f\}$.

Theorem

R *has perspectivity transitive iff for all* $e, f \in E(R)$, $e \sim_{r\ell r} f \Rightarrow e \sim_{\ell r\ell} f$.

Two “interesting” IC rings

Let $D = \mathbb{Z}$ and $S = T^{-1}D$ the localisation of D in T, where T is the multiplicative close of prime numbers p such that $p \equiv \pm 1 (\mathrm{mod}\ 8)$.

1. S has not stable range 1;

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1. S has not stable range 1;
2. We form the two matrix rings

$$R_2 = \begin{pmatrix} S & 2S \\ 2S & S \end{pmatrix} \text{ and } R_4 = \begin{pmatrix} S & 4S \\ 4S & S \end{pmatrix}.$$

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- Using Dirichlet's theorem on prime numbers in arithmetic progression, we can prove that R_2 is $3/2$-perspective and that R_4 has perspectivity transitive.

$3/2$-perspective rings are actually abundant

Theorem

1. *Let S be any nontrivial localization of $\mathbb{Z}$. Then under a Generalized Riemann Hypothesis, $\mathcal{M}_2(S)$ satisfies $\mathcal{P}(4)$ (R is $3/2$-perspective, all its regular elements special clean) (and $\mathcal{P}(5)$ without GRH);*

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2. *Let S be a projective-free ring with $n \geq 2$ in its stable range. If $m \geq 4n-5$, then $R = \mathcal{M}_m(S)$ satisfies $\mathcal{P}(4)$. If S has not stable range 1, R is not perspective;*

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3. *For instance $\mathcal{M}_m(\mathbb{Z})$ is $3/2$-perspective but not perspective for all $m \geq 3$.*

Thank you for your attention.

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